Power Control for Block-Fading Channels with Peak-to-Average Power Constraints

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Abstract— Power allocation with peak-to-average power constraints over Nakagami-m block-fading channels with arbitrary input distributions is studied. In particular, we find the solution to the minimum outage power allocation scheme with peak-to-average power constraints and arbitrary input distributions, and show that the signal-to-noise ratio exponent for any systems with a finite peak-to-average power ratio constraint is the same as that of systems with peak-power constraints, resulting in an error floor.

I. INTRODUCTION

One major design challenge in wireless communication is to cope with the varying nature of the channel. In practical systems, codewords are transmitted over channels with a finite number of degrees of freedom. Examples for such scenarios are transmission over slowly varying channels with codeword length larger than the channel bandwidth-time coherence product, or transmission using OFDM over frequency-selective channels. The block-fading channel, introduced in [1], [2], is a mathematical channel model that captures the features of these scenarios. In this model, a codeword is transmitted over B flat fading blocks, where the B channel fading gains vary according to a system-specific fading distribution.

Since each codeword experiences a finite number B of fading realizations, the largest achievable rate is a random variable. For most practical fading statistics, the channel capacity is zero since there is a non-zero probability that any positive communication rate is not supported by the channel. A relevant performance measure in this case is the information outage probability, which is the probability that communication at a rate R is not supported by the channel [1], [2]. This probability is a lower bound on the word error rate over the channel [3].

When knowledge of the channel fading coefficients, also known as channel state information (CSI), is available at the transmitter, power allocation schemes can be employed to minimize the outage probability. Transmitter CSI can be obtained from a dedicated feedback channel or by reusing the receiver CSI for transmission in time-division duplex (TDD) systems [4]. The optimal power allocation problem has been investigated in [3] for channels with Gaussian inputs, and in [5], [6] for channels with arbitrary input constellations. The

works in [3], [6] consider systems with peak (per-codeword) and average power constraints, and show that systems with average constraints perform significantly better than systems with peak power constraints. However, systems with average constraints employ very large (possibly infinite) peak power, which is not feasible in practice.

In this work, we consider power allocation for systems with both peak and average power constraints. This problem has been solved for systems with Gaussian inputs in [3]. We extend the results to systems with arbitrary input constellations. We show that for a system with fixed peak-to-average power ratio (PAPR) and a large peak power constraint, subject to Nakagami-m fading statistics, the outage performance at large signal-to-noise ratios is the same as that of systems with peak power constraint only.

The remainder of the paper is organized as follows. Sections II and III describe the system model and the information theoretic framework of the work. Section IV reviews the optimal power allocation schemes for systems with peak and average power constraints. Performance of systems with peak-to-average power ratio constraints, which is also the main contribution of the paper, is discussed in Section V. Finally, concluding remarks are given in Section VI.

II. SYSTEM MODEL

Consider transmission over a channel consisting of B blocks of L channel uses, in which, block $b, b = 1, \ldots, B$ undergoes an independent fading gain h_b, corresponding to a power fading gain $\gamma_b \triangleq |h_b|^2$. Assume that the power fading gains $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_B)$ are available at both the transmitter and the receiver. Suppose that the transmit power is allocated following the rule $\boldsymbol{p}(\boldsymbol{\gamma}) = (p_1(\boldsymbol{\gamma}), \ldots, p_B(\boldsymbol{\gamma}))$. Then the corresponding complex baseband channel model is given by

$$\boldsymbol{y}_b = \sqrt{p_b(\boldsymbol{\gamma})} h_b \boldsymbol{x}_b + \boldsymbol{z}_b, \ b = 1, \ldots, B, \qquad (1)$$

where $\boldsymbol{y}_b \in \mathbb{C}^L$ and $\boldsymbol{x}_b \in \mathcal{X}^L$, with $\mathcal{X} \subset \mathbb{C}$ being the signal constellation set, are the received and transmitted signals in block b, respectively, and $\boldsymbol{z}_b \in \mathbb{C}^L$ is the noise vector with independently identically distributed circularly symmetric Gaussian entries $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$. Assume that the signal constellation $\mathcal{X}$ of size 2^M satisfies $\sum_{x \in \mathcal{X}} |x|^2 = 2^M$, then the

instantaneous received SNR at block b is given by $p_b(\boldsymbol{\gamma})\gamma_b$. We consider systems with the following power constraints:

Average power: $\quad \mathbb{E}\left[\langle\boldsymbol{p}(\boldsymbol{\gamma})\rangle\right] \leq P_{\text{av}}$

Peak power: $\quad \langle\boldsymbol{p}(\boldsymbol{\gamma})\rangle \triangleq \frac{1}{B}\sum_{b=1}^{B}p_b(\boldsymbol{\gamma}) \leq P_{\text{peak}}$.

Power allocation schemes for systems with peak and average power constraints has been studied in [3], [6][2]. Power allocation with average power constraints offers large performance advantages but may require large peak powers [6], [3]. Large peak powers, however, may prohibit application in practical systems. In this work, we study the performance of systems with peak power constraints in addition to average power constraints [3]. In particular, we consider systems with a constrained *peak-to-average power ratio*

$$\text{PAPR} \triangleq \frac{P_{\text{peak}}}{P_{\text{av}}} \geq 1.$$

For block b, the magnitude $|h_b|$ of the fading gain is assumed to follow a Nakagami-m distribution

$$f_{|h_b|}(\xi) = \frac{2m^m\xi^{2m-1}}{\Gamma(m)}e^{-m\xi^2}, \quad \xi \geq 0,$$

where $\Gamma(a)$ is the Gamma function, $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$. The pdf of the power fading gain is then given by

$$f_{\gamma_b}(\gamma) = \begin{cases} \frac{m^m\gamma^{m-1}}{\Gamma(m)}e^{-m\gamma}, & \gamma \geq 0 \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

The Nakagami-m distribution represents a large class of practical fading statistics. In particular, we can recover the Rayleigh fading by setting $m = 1$ and approximate the Rician fading with parameter K by setting $m = \frac{(K+1)^2}{2K+1}$ [7].

III. MUTUAL INFORMATION AND OUTAGE PROBABILITY

Given a channel realization $\boldsymbol{\gamma}$ and a power allocation scheme $\boldsymbol{p}(\boldsymbol{\gamma})$, the instantaneous input-output mutual information of the block-fading channel given in (1) is given by

$$I_B(\boldsymbol{p}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) = \frac{1}{B}\sum_{b=1}^{B}I_{\mathcal{X}}(p_b\gamma_b),$$

where $I_{\mathcal{X}}(\rho)$ is the input-output mutual information of the channel with input constellation $\mathcal{X}$ and received SNR ρ. With Gaussian inputs, we have that $I_{\mathcal{X}_G}(\rho) = \log_2(1+\rho)$, while for coded modulation over uniformly distributed discrete signal constellations, we have that

$$I_{\mathcal{X}}(\rho)$$
$$= M - \frac{1}{2^M}\sum_{x\in\mathcal{X}}\mathbb{E}_Z\left[\log_2\left(\sum_{x'\in\mathcal{X}}e^{-|\sqrt{\rho}(x-x')+Z|^2+|Z|^2}\right)\right],$$

where the expectation is over $Z \sim \mathcal{N}_{\mathbb{C}}(0,1)$. We also consider systems with bit-interleaved coded modulation (BICM)

using the classical non-iterative BICM decoder proposed by Zehavi in [8]. The input-output mutual information for a given labelling rule can be expressed as [9]

$$I_{\mathcal{X}}^{\text{BICM}}(\rho) =$$
$$M - \frac{1}{2^M}\sum_{c=0}^{1}\sum_{j=1}^{M}\sum_{x\in\mathcal{X}_c^j}\mathbb{E}_Z\left[\log_2\frac{\sum_{x'\in\mathcal{X}}e^{-|\sqrt{\rho}(x-x')+Z^2|}}{\sum_{x'\in\mathcal{X}_c^j}e^{-|\sqrt{\rho}(x-x')+Z^2|}}\right],$$

where the sets $\mathcal{X}_c^j$ contain all signal constellation points with bit c in the j-th binary labelling position.

For a fixed transmission rate R, communication is in outage whenever $I_B(\boldsymbol{p}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) < R$, and the outage probability, which is a lower bound to the word error probability, is given by

$$P_{\text{out}}\left(\boldsymbol{p}(\boldsymbol{\gamma}), P, R\right) \triangleq \Pr\left(I_B(\boldsymbol{p}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) < R\right).$$

Using the channel knowledge at the transmitter and receiver, the power allocation scheme $\boldsymbol{p}(\boldsymbol{\gamma})$ aims at minimizing the outage probability, improving system performance.

In deriving the optimal power allocation schemes, a useful measure is the first derivative of the input-output mutual information $I_{\mathcal{X}}(\rho)$ with respect to the signal-to-noise ratio [5], [6]. From [10] we have that,

$$\frac{d}{d\rho}I_{\mathcal{X}}(\rho) = \frac{1}{\log 2}\text{MMSE}_{\mathcal{X}}(\rho),$$

where $\text{MMSE}_{\mathcal{X}}(\rho)$ is the minimum mean squared error (MMSE) in estimating an input symbol in $\mathcal{X}$ transmitted over an AWGN channel with SNR ρ. For Gaussian inputs, $\text{MMSE}_{\mathcal{X}_G}(\rho) = \frac{1}{1+\rho}$, while for general constellation $\mathcal{X}$, we have that [5]

$$\text{MMSE}_{\mathcal{X}}(\rho) = \frac{1}{2^M}\sum_{x\in\mathcal{X}}|x|^2 - \frac{1}{\pi}\int_{\mathbb{C}}\frac{\left|\sum_{x\in\mathcal{X}}xe^{-|y-\sqrt{\rho}x|^2}\right|^2}{\sum_{x\in\mathcal{X}}e^{-|y-\sqrt{\rho}x|^2}}dy.$$

For systems with BICM, the first derivative of the input-output mutual information with respect to signal-to-noise ratio is given by [11][3]

$$\frac{d}{d\rho}I_{\mathcal{X}}^{\text{BICM}}(\rho) = \frac{1}{2\log 2}\sum_{j=1}^{M}\sum_{c=0}^{1}\left(\text{MMSE}_{\mathcal{X}}(\rho) - \text{MMSE}_{\mathcal{X}_c^j}(\rho)\right)$$
$$\triangleq \text{MMSE}_{\mathcal{X}}^{\text{BICM}}(\rho).$$

In the remainder of the paper, we perform our analysis for the coded modulation case. Results for the BICM case can be obtained by simply replacing $I_{\mathcal{X}}(\rho), \text{MMSE}_{\mathcal{X}}(\rho)$ by $I_{\mathcal{X}}^{\text{BICM}}(\rho)$ and $\text{MMSE}_{\mathcal{X}}^{\text{BICM}}(\rho)$, respectively.

IV. PEAK AND AVERAGE POWER CONSTRAINTS

In this section, we review known results on peak and average power constrained systems that will be used for our main results. A detailed treatment of optimal and suboptimal power allocation schemes for systems with peak and average power constraints is given in [6].

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[2]In the literature, peak power constraints have also been referred to as short-term power constraints, and average power constraints as long-term power constraints.

[3]With some abuse of notation, we use $\text{MMSE}_{\mathcal{X}}^{\text{BICM}}(\rho)$ to denote the first derivative with respect to ρ of the mutual information. However, $\text{MMSE}_{\mathcal{X}}^{\text{BICM}}(\rho)$ is not the minimum mean square error in estimating the channel input from its output, since the noise is not Gaussian due to the demodulation process.

A. Peak Power Constraint

For systems with peak power constraint P_{peak}, the optimal power allocation scheme is the solution of the following problem [3]

$$
\begin{cases}
\text{Minimize} & P_{\text{out}}\left(\boldsymbol{p}(\boldsymbol{\gamma}), P_{\text{peak}}, R\right) \\
\text{Subject to} & \langle \boldsymbol{p}(\boldsymbol{\gamma}) \rangle \leq P_{\text{peak}} \\
& p_b \geq 0, \ b = 1, \ldots, B
\end{cases}
\tag{3}
$$

The solution is given by [5], [6]

$$
p_b^{\text{peak}}(\boldsymbol{\gamma}) = \frac{1}{\gamma_b} \text{MMSE}_{\mathcal{X}}^{-1}\left(\min\left\{\text{MMSE}_{\mathcal{X}}(0), \frac{\eta}{\gamma_b}\right\}\right), \tag{4}
$$

for $b = 1, \ldots, B$, where η is chosen such that the peak power constraint is met with equality. As shown in [6], one can alternatively find the optimal power allocation with peak power constraint by

$$
\boldsymbol{p}^{\text{peak}}(\boldsymbol{\gamma}) = \begin{cases}
\wp(\boldsymbol{\gamma}), & \text{if } \langle \wp(\boldsymbol{\gamma}) \rangle \leq P_{\text{peak}} \\
\boldsymbol{0}, & \text{otherwise},
\end{cases}
\tag{5}
$$

where $\wp(\boldsymbol{\gamma})$ is the solution of the problem

$$
\begin{cases}
\text{Minimize} & \langle \wp(\boldsymbol{\gamma}) \rangle \\
\text{Subject to} & I_B(\wp(\boldsymbol{\gamma}), \boldsymbol{\gamma}) \geq R \\
& \wp_b \geq 0, \ b = 1, \ldots, B.
\end{cases}
\tag{6}
$$

From [6], $\wp(\boldsymbol{\gamma})$ is given by

$$
\wp_b(\boldsymbol{\gamma}) = \frac{1}{\gamma_b} \text{MMSE}_{\mathcal{X}}^{-1}\left(\min\left\{\text{MMSE}_{\mathcal{X}}(0), \frac{\eta}{\gamma_b}\right\}\right), \tag{7}
$$

for $b = 1, \ldots, B$, where η is now chosen such that the rate constraint is met,

$$
\frac{1}{B}\sum_{b=1}^{B} I_{\mathcal{X}}\left(\text{MMSE}_{\mathcal{X}}^{-1}\left(\min\left\{\text{MMSE}_{\mathcal{X}}(0), \frac{\eta}{\gamma_b}\right\}\right)\right) = R.
$$

The minimum outage probability is given by

$$
P_{\text{out}}\left(\boldsymbol{p}^{\text{peak}}, P_{\text{peak}}, R\right), R = \Pr(\langle \wp(\boldsymbol{\gamma}) \rangle > P_{\text{peak}}). \tag{8}
$$

The power allocation scheme given in (4) is less complex than the one given in (5) for systems with peak power constraints. However, the two schemes are equivalent in terms of outage probability, and the latter is useful for the analysis of systems with average and peak-to-average power ratio constraints. In the following, we only refer to $\boldsymbol{p}^{\text{peak}}(\boldsymbol{\gamma})$ given in (5) when considering systems with peak power constraints.

B. Average Power Constraint

Under average power constraint, the optimal power allocation scheme solves the following problem

$$
\begin{cases}
\text{Minimize} & \Pr(I_B(\boldsymbol{p}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) < R) \\
\text{Subject to} & \mathbb{E}\left[\langle \boldsymbol{p}(\boldsymbol{\gamma}) \rangle\right] \leq P_{\text{av}}.
\end{cases}
\tag{9}
$$

From [6], the solution $\boldsymbol{p}^{\text{av}}(\boldsymbol{\gamma})$ of (9) is given by

$$
\boldsymbol{p}^{\text{av}}(\boldsymbol{\gamma}) = \begin{cases}
\wp(\boldsymbol{\gamma}), & \langle \wp(\boldsymbol{\gamma}) \rangle \leq s \\
\boldsymbol{0}, & \text{otherwise},
\end{cases}
\tag{10}
$$

where $\wp(\boldsymbol{\gamma})$ is the scheme minimizing the peak power given in (7) and s is such that

$$
\begin{cases}
s = \infty, & \text{if } \lim_{s \to \infty} \mathbb{E}\left[\langle \boldsymbol{p}^{\text{av}}(\boldsymbol{\gamma}) \rangle\right] \leq P_{\text{av}} \\
P_{\text{av}} = \mathbb{E}\left[\langle \boldsymbol{p}^{\text{av}}(\boldsymbol{\gamma}) \rangle\right], & \text{otherwise},
\end{cases}
\tag{11}
$$

The threshold s is a function of $\wp(\boldsymbol{\gamma})$, P_{av} and the fading statistics $f_{\gamma_b}(\boldsymbol{\gamma})$, thus s is fixed and can be predetermined. The minimum outage probability is given by

$$
P_{\text{out}}\left(\boldsymbol{p}^{\text{av}}(\boldsymbol{\gamma}), P_{\text{av}}, R\right) = \Pr(\langle \wp(\boldsymbol{\gamma}) \rangle > s). \tag{12}
$$

V. Peak-to-Average Power Ratio Constraints

Following [3], for systems with average power P_{av} and peak-to-average power ratio PAPR, the optimal power allocation scheme solves the following problem

$$
\begin{cases}
\text{Minimize} & \Pr\left(I_B(\boldsymbol{p}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) < R\right) \\
\text{Subject to} & \langle \boldsymbol{p}(\boldsymbol{\gamma}) \rangle \leq P_{\text{peak}} \\
& \mathbb{E}\left[\langle \boldsymbol{p}(\boldsymbol{\gamma}) \rangle\right] \leq P_{\text{av}},
\end{cases}
\tag{13}
$$

where $P_{\text{peak}} = \text{PAPR} \cdot P_{\text{av}}$.

Following the arguments in [3], we have the following result

Proposition 1: A solution to problem (13) is given by

$$
\boldsymbol{p}^{\star}(\boldsymbol{\gamma}) = \begin{cases}
\wp(\boldsymbol{\gamma}), & \langle \wp(\boldsymbol{\gamma}) \rangle \leq \hat{s} \\
\boldsymbol{0}, & \text{otherwise},
\end{cases}
\tag{14}
$$

where $\wp(\boldsymbol{\gamma})$ is given in (7) and $\hat{s} = \min\{s, P_{\text{peak}}\}$ with s defined as in (11).

Proof: If $s \leq P_{\text{peak}}$, then $\langle \boldsymbol{p}^{\star}(\boldsymbol{\gamma}) \rangle \leq s \leq P_{\text{peak}}$. Thus, the peak power constraint is satisfied. Additionally, since $\hat{s} = s$, $\boldsymbol{p}^{\star}(\boldsymbol{\gamma})$ coincides with the optimal power allocation scheme $\boldsymbol{p}^{\text{av}}(\boldsymbol{\gamma})$ in (10) for system with average power constraint P_{av}. Therefore, $\boldsymbol{p}^{\star}(\boldsymbol{\gamma})$ is a solution for the problem in (13).

If $s > P_{\text{peak}}$, then

$$
\boldsymbol{p}^{\star}(\boldsymbol{\gamma}) = \begin{cases}
\wp(\boldsymbol{\gamma}), & \langle \wp(\boldsymbol{\gamma}) \rangle < P_{\text{peak}} < s \\
\boldsymbol{0}, & \text{otherwise}.
\end{cases}
\tag{15}
$$

Therefore, $\mathbb{E}\left[\langle \boldsymbol{p}^{\star}(\boldsymbol{\gamma}) \rangle\right] < \mathbb{E}\left[\langle \boldsymbol{p}^{\text{av}}(\boldsymbol{\gamma}) \rangle\right] \leq P_{\text{av}}$, and thus, the average power constraint is satisfied. Additionally, $\boldsymbol{p}^{\star}(\boldsymbol{\gamma})$ in (15) is also an optimal power allocation scheme for systems with peak power constraint P_{peak} given in (5). Thus, $\boldsymbol{p}^{\star}(\boldsymbol{\gamma})$ is a solution for the problem in (13). $\blacksquare$

Remark 1: From the proof, depending on P_{av} and the PAPR (which is fixed), one of the power constraints is redundant and the outage performance is dependent on the remaining constraint. In particular we have that

- if $s > P_{\text{peak}}$,

$$
P_{\text{out}}\left(\boldsymbol{p}^{\star}(\boldsymbol{\gamma}), P_{\text{av}}, R\right) = P_{\text{out}}\left(\boldsymbol{p}^{\text{peak}}(\boldsymbol{\gamma}), P_{\text{peak}}, R\right).
$$

- if $s \leq P_{\text{peak}}$,

$$
P_{\text{out}}\left(\boldsymbol{p}^{\star}(\boldsymbol{\gamma}), P_{\text{av}}, R\right) = P_{\text{out}}\left(\boldsymbol{p}^{\text{av}}(\boldsymbol{\gamma}), P_{\text{av}}, R\right).
$$

Consequently, the outage probability can also be evaluated as

$$P_{\text{out}}\left(\boldsymbol{p}^{\star}(\boldsymbol{\gamma}), P_{\text{av}}, R\right) =$$
$$\max\left\{P_{\text{out}}\left(\boldsymbol{p}^{\text{peak}}(\boldsymbol{\gamma}), P_{\text{peak}}, R\right), P_{\text{out}}\left(\boldsymbol{p}^{\text{av}}(\boldsymbol{\gamma}), P_{\text{av}}, R\right)\right\} =$$
$$\max\left\{P_{\text{out}}\left(\boldsymbol{p}^{\text{peak}}(\boldsymbol{\gamma}), \text{PAPR}\, P_{\text{av}}, R\right), P_{\text{out}}\left(\boldsymbol{p}^{\text{av}}(\boldsymbol{\gamma}), P_{\text{av}}, R\right)\right\}.$$
$$(16)$$

The above expression clearly highlights that in order to compute the outage probability with peak-to-average power ratio constraints, it is sufficient to translate the curve corresponding to the peak power constraint by PAPR dB and then find the maximum between the translated curve and the curve corresponding to the average power constraint.

A. Asymptotic Analysis

In this section we study the asymptotic behavior of the outage probability under peak-to-average power constraints. In particular, we study the SNR exponents, i.e., the asymptotic slope of the outage probability for large SNR. For large P_{av}, we have the following result.

Proposition 2: Consider transmission at rate R over the block-fading channel given in (1) with power allocation scheme $\boldsymbol{p}^{\star}(\boldsymbol{\gamma})$. Assume input constellation $\mathcal{X}$ of size 2^M. Further assume that the power fading gains $\boldsymbol{\gamma}$ follow the Nakagami-m distribution given in (2). Then, at large P_{av} and any PAPR $< \infty$, the outage probability behaves like

$$P_{\text{out}}\left(\boldsymbol{p}^{\star}(\boldsymbol{\gamma}), P_{\text{av}}, R\right) \doteq \mathcal{K} P_{\text{av}}^{-md(R)} \qquad (17)$$

where $d(R)$ is the Singleton bound [12], [13], [14], [15]

$$d(R) = 1 + \left\lfloor B\left(1 - \frac{R}{M}\right)\right\rfloor.$$

Proof: First, for very large P_{peak} we have that [6],

$$P_{\text{out}}\left(\boldsymbol{p}^{\text{peak}}(\boldsymbol{\gamma}), P_{\text{peak}}, R\right) \doteq \mathcal{K}^{\text{peak}} P_{\text{peak}}^{-md(R)}. \qquad (18)$$

Therefore, denote $P(s)$ as the average power as a function of the threshold s in the allocation scheme $\boldsymbol{p}^{\text{av}}(\boldsymbol{\gamma})$ in (10). Asymptotically with s, we have [6]

$$\frac{d}{ds}P(s) \doteq \mathcal{K}^{\text{peak}} d(R) s^{-d(R)}.$$

From L'Hôpital's rule, for any PAPR, we have

$$\lim_{s\to\infty} \frac{\text{PAPR} \cdot P(s)}{s} = \lim_{s\to\infty} \frac{d}{ds}\text{PAPR} \cdot P(s)$$
$$= \lim_{s\to\infty} \text{PAPR} \cdot \mathcal{K} d(R) s^{-d(R)}$$
$$= 0.$$

It follows that for any PAPR, there exists an average power constraint $P_0 = P(s_0)$ satisfying $s_0 = \text{PAPR} \cdot P_0$ and $s > \text{PAPR} \cdot P_{\text{av}} = P_{\text{peak}}$ if $P(s) > P_0$. Consequently, $P_{\text{out}}\left(\boldsymbol{p}^{\star}(\boldsymbol{\gamma}), P_{\text{av}}, R\right) = P_{\text{out}}\left(\boldsymbol{p}^{\text{peak}}(\boldsymbol{\gamma}), \text{PAPR} \cdot P_{\text{av}}, R\right)$ for $P_{\text{av}} > P_0$. Thus, together with (18), asymptotically in P_{av}, we have

$$P_{\text{out}}\left(\boldsymbol{p}^{\star}(\boldsymbol{\gamma}), P_{\text{av}}, R\right) \doteq P_{\text{out}}\left(\boldsymbol{p}^{\text{peak}}(\boldsymbol{\gamma}), \text{PAPR} \cdot P_{\text{av}}, R\right)$$
$$\doteq \mathcal{K}^{\text{peak}} \text{PAPR}^{-md(R)} P_{\text{av}}^{-md(R)}$$

as stated in (17). ∎

Let P_0 be the largest root of the following equation

$$\mathbb{E}_{\mathcal{R}(P_0)}\left[\langle \wp(\boldsymbol{\gamma})\rangle\right] = \text{PAPR} \cdot P_0, \qquad (19)$$

we therefore have that

$$P_{\text{out}}\left(\boldsymbol{p}^{\star}(\boldsymbol{\gamma}), P_{\text{av}}, R\right) = P_{\text{out}}\left(\boldsymbol{p}^{\text{peak}}(\boldsymbol{\gamma}), \text{PAPR} \cdot P_{\text{av}}, R\right)$$

for $P_{\text{av}} > P_0$. Therefore, for asymptotically large P_{av}, the outage probability for systems with a PAPR constraint is determined by the outage probability of systems with peak power constraint $P_{\text{peak}} = \text{PAPR} \cdot P_{\text{av}}$. As a consequence of the above analysis, we have that the delay-limited capacity [16] is zero for any finite PAPR. This is illustrated by the examples in the next section.

B. Numerical Results

For simplicity, we first consider the outage performance of systems with $B = 1$ under Nakagami-m fading statistic. In this case, the outage probability can be numerically evaluated as follows. Let γ be the power fading gain, then

$$\wp(\gamma) = \frac{I_{\mathcal{X}}^{-1}(R)}{\gamma},$$

and (19) is equivalent to

$$\int_{\frac{I_{\mathcal{X}}^{-1}(R)}{P_1}}^{\infty} \frac{I_{\mathcal{X}}^{-1}(R)}{\gamma} \frac{m^m \gamma^{m-1}}{\Gamma(m)} e^{-m\gamma} d\gamma = \text{PAPR} \cdot P_1$$

$$\frac{m^m}{\Gamma(m)} a \int_a^{\infty} \gamma^{m-2} e^{-m\gamma} d\gamma = \text{PAPR}$$

$$\frac{m}{\Gamma(m)} a \Gamma(m-1, ma) = \text{PAPR},$$

where $P_1 \triangleq \frac{I_{\mathcal{X}}^{-1}(R)}{a}$ and $\Gamma(n, \xi)$ is the upper incomplete Gamma function [17] $\Gamma(n, \xi) = \int_{\xi}^{\infty} t^{n-1} e^{-t} dt$. For $P > P_1$ ($s > \text{PAPR} \cdot P$) the outage probability is given by

$$P_{\text{out}}\left(\boldsymbol{p}^{\text{peak}}(\boldsymbol{\gamma}), \text{PAPR} \cdot P, R\right) = \Pr\left(\gamma < \frac{I_{\mathcal{X}}^{-1}(R)}{\text{PAPR} \cdot P}\right)$$
$$= F_{\gamma}\left(\frac{I_{\mathcal{X}}^{-1}(R)}{\text{PAPR} \cdot P}\right),$$

where $F_{\gamma}(\xi) = \Pr(\gamma \leq \xi)$ is the cumulative distribution function of γ. For $P < P_1$ ($s < \text{PAPR} \cdot P$), s in (10) is obtained by solving

$$\frac{m I_{\mathcal{X}}^{-1}(R)}{\Gamma(m)} \Gamma\left(m-1, \frac{m I_{\mathcal{X}}^{-1}(R)}{s}\right) = P,$$

and the outage probability is given by

$$P_{\text{out}}\left(\boldsymbol{p}^{\text{av}}(\boldsymbol{\gamma}), P, R\right) = \Pr\left(\gamma < \frac{I_{\mathcal{X}}^{-1}(R)}{s}\right)$$
$$= F_{\gamma}\left(\frac{I_{\mathcal{X}}^{-1}(R)}{s}\right).$$

The analysis result is illustrated in Figure 1 for 16-QAM input Rayleigh faded channel at rate $R = 1$. We observe that as we increase the PAPR, the error floor occurs at lower error
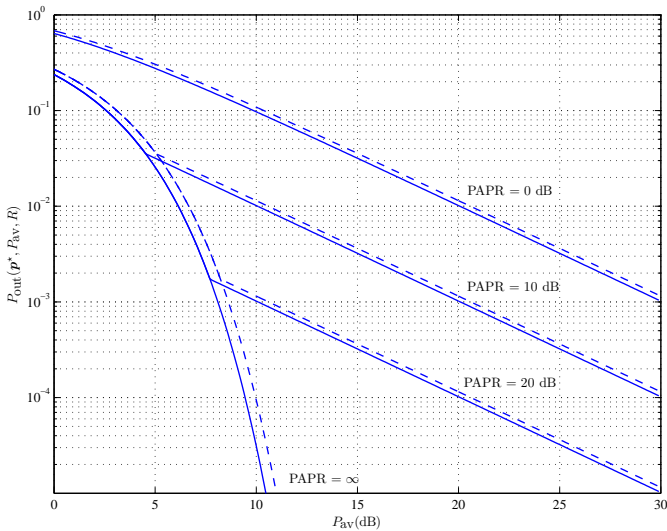
Fig. 1. Outage probability for systems with PAPR constraints over Nakagami-m block-fading channels $B = 1, m = 1, R = 1$, 16-QAM inputs. The solid and dashed lines correspondingly represent outage probability of systems with coded modulation and BICM.

Fig. 2. Outage probability for systems with PAPR constraints over Nakagami-m block-fading channels $B = 4, m = 1, R = 3$, 16-QAM inputs. The solid and dashed lines correspondingly represent outage probability of systems with coded modulation and BICM.

probability values, and eventually, at values below a target quality of service error rate. We also observe that the loss incurred by BICM is minimal.

For systems with any $B > 1$, analytical result is not available. However, from (16), the outage probability of systems with PAPR constraints can be obtained by considering systems with peak power constraints and systems with average power constraints separately. Moreover, at high P_{av}, the outage probability can be obtained by the outage probability of systems with only peak power constraint $P_{\mathrm{av}} \cdot \mathrm{PAPR}$. Simulation results for a 16-QAM input, Rayleigh faded channel with 4 blocks at rate $R = 3$ using are given in Figure 2.

In both cases ($B = 1$ and $B = 4$), at high P_{av}, the outage probability given by optimal power allocation scheme is governed by the peak power constraints, and therefore, the optimal outage diversity is given by the Singleton bound.

VI. CONCLUSIONS

We have studied power allocation schemes under peak-to-average power constraints for block-fading channels with arbitrary input distributions. We have computed the optimal solution, and shown that the resulting outage probability can easily be computed from the corresponding solutions with peak and average power constraints. We have studied the SNR exponents, and have shown that the asymptotic performance for finite PAPR is always determined by the peak power, and that the exponent is therefore given by the exponent of systems with peak power constraints.

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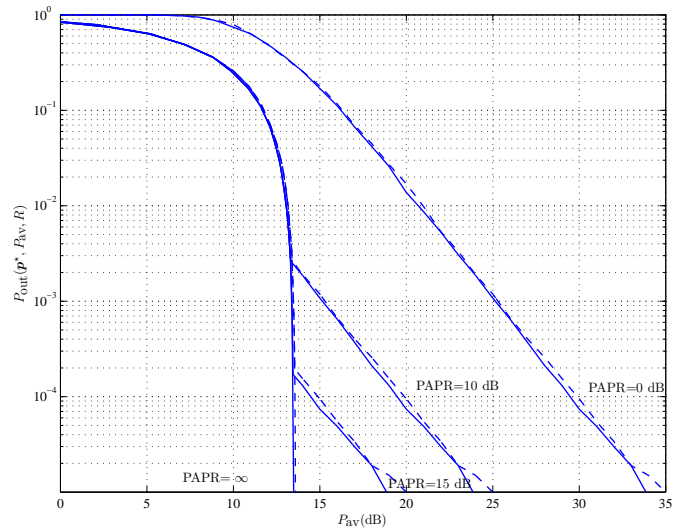
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