Asymptotic Outage Performance of Power Allocation in Block-Fading Channels

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Abstract— We characterize the asymptotic outage performance of power allocation techniques for systems with average power constraints. We show that the outage diversity of a system with average power constraints can be obtained from the diversity of the corresponding system with peak power constraints. The characterization is therefore useful since the asymptotic performance of systems with peak power constraints is well known in the literature.

I. INTRODUCTION

In many practical communication systems, codewords of fixed rate are to be delivered over slowly time/frequency-varying fading channels under stringent delay constraints. Examples of such cases are Orthogonal Frequency Division Multiplexing (OFDM) or slow frequency-hopping systems used for fixed-rate transmission over slowly varying channels. The block-fading channel [1], [2] is useful for the study of such communication scenarios. In this model, fixed-rate codewords are transmitted within a finite number of blocks, each of which is affected by an independent fading coefficient. Therefore, each codeword experiences a finite number of degrees of freedom. Consequently, for most fading statistics, there exists a non-zero probability that the channel does not support the desired data rate, also known as the information outage probability [1], [2]. The information outage probability is also the word error rate given that an optimal coding scheme is employed for transmission [3]. Therefore, the channel has zero capacity and the outage probability is the relevant performance measure.

One system design criterion of importance is therefore to minimize the outage probability. When channel state information is available at the transmitter, power allocation, among other adaptive transmission schemes, can be used to reduce the outage probability. The optimal power allocation scheme for systems with Gaussian inputs is derived in [3], [4]. Optimal and suboptimal schemes for systems with arbitrary inputs are also proposed in [5], [6]. These works consider power allocation for systems with peak (per codeword) and average power constraints, and show that power control for average power constraints provides significant performance gains compared to that for peak power constraints. Moreover,

for a specific class of fading statistics, power control for average power constraints may give zero outage, yielding a non-zero delay-limited capacity [4].

In this work, we perform an asymptotic performance analysis of systems with average power constraints and Nakagami-m fading statistics. We show the duality between the power allocation rules resulting from peak and average power constraints. This duality is instrumental in characterizing the asymptotic performance of systems with average power constraints based on that of systems with peak constraints. The result can be applied to any system whose performance at large peak power constraints is known. This is a useful result since the asymptotic behavior of systems with peak power constraints is obtainable, e.g., in [7], [8], [9], [10], [6].

The remainder of the paper is organized as follows. The system model and the information theoretic base of the work is given in Sections II and III. Sections IV and V discuss the power allocation schemes for systems with peak and average power constraints and their asymptotic performance. Finally, concluding remarks are given in Section VI.

The following notations are used throughout the paper. We denote $\langle \boldsymbol{x} \rangle = \frac{1}{B} \sum_{i=1}^{B} x_i$ as the arithmetic mean of $\boldsymbol{x} = (x_1, \ldots, x_B)$ and $\mathbb{E}[\cdot]$ as the expected value of a random variable. The exponent equality $f(\xi) \doteq \mathcal{K}\xi^{-d}$ indicates that $\lim_{\xi \to \infty} f(\xi)\xi^d = \mathcal{K}$, with the exponential inequalities $\dot{\leq}, \dot{\geq}$ similarly defined. $\lfloor \xi \rfloor$ denotes the largest integer not greater than ξ. Component-wise inequalities between two vectors are denoted by $\succeq$ and $\preceq$. Finally, $\mathbb{1}_{\mathcal{A}}$ takes value 1 if the event $\mathcal{A}$ is true and takes value 0 otherwise.

II. SYSTEM MODEL

Consider transmission over a block-fading channel with B blocks, where each is affected by a flat fading coefficient h_b, $b = 1, \ldots, B$ and additive noise. Assume that the fading coefficients are available at both the transmitter and the receiver, and that the transmitter allocates power to the blocks according to the rule $\boldsymbol{p}(\boldsymbol{\gamma}) = (p_1(\boldsymbol{\gamma}), \ldots, p_B(\boldsymbol{\gamma}))$ where $\boldsymbol{\gamma}$ is the power fading gain vector defined as $\gamma_b = |h_b|^2, b = 1, \ldots, B$. The equivalent baseband model is given by

$$\boldsymbol{y}_b = \sqrt{p_b(\boldsymbol{\gamma})} h_b \boldsymbol{x}_b + \boldsymbol{n}_b, \quad b = 1, \ldots, B, \qquad (1)$$

where $\boldsymbol{x}_b \in \mathbb{C}^L$ and $\boldsymbol{y}_b \in \mathbb{C}^L$ are correspondingly the portion of the codeword transmitted and received in block b. Assume that $\boldsymbol{n}_b \in \mathbb{C}^L$ is a white Gaussian noise vector with unit variance and $\boldsymbol{x}_b$ is drawn from a unit-energy constellation $\mathcal{X}$, $\mathbb{E}_{x \in \mathcal{X}}[x] = 1$. Then, the instantaneous received signal-to-noise ratio (SNR) at block b is given by $p_b(\boldsymbol{\gamma})\gamma_b$. We consider systems with the following power constraints:

$$\text{Peak}: \quad \langle \boldsymbol{p}(\boldsymbol{\gamma}) \rangle \leq P_{\text{peak}}$$
$$\text{Average}: \quad \mathbb{E}\left[\langle \boldsymbol{p}(\boldsymbol{\gamma}) \rangle\right] \leq P_{\text{av}}$$

We assume that the fading gains h_b are independently identically distributed random variables whose magnitude follows the Nakagami-m distribution with unit mean[2]. The probability density function (pdf) of $|h_b|$ is then given by

$$f_{|h_b|}(\xi) = \frac{2m^m \xi^{2m-1}}{\Gamma(m)} e^{-m\xi^2}, \ \xi \geq 0$$

where $\Gamma(m)$ is the Gamma function given by $\Gamma(a) = \int_0^\infty t^{a-1} dt$. The pdf of the power fading gain γ_b is given by

$$f_{\gamma_b}(\xi) = \frac{m^m \xi^{m-1}}{\Gamma(m)} e^{-m\xi}, \ \xi \geq 0. \tag{2}$$

The Nakagami-m distribution represents a large class of fading statistics, including the Rayleigh fading by letting $m = 1$, and an approximation of the Ricean fading with parameter K by setting $m = \frac{(K+1)^2}{2K+1}$ [11].

III. MUTUAL INFORMATION AND OUTAGE PROBABILITY

For a given fading realization $\boldsymbol{\gamma}$ and power allocation scheme $\boldsymbol{p}(\boldsymbol{\gamma})$, the instantaneous mutual information is

$$I_B(\boldsymbol{p}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) \triangleq \frac{1}{B} \sum_{b=1}^{B} I_{\mathcal{X}}(p_b(\boldsymbol{\gamma})\gamma_b),$$

where $I_{\mathcal{X}}(\rho)$ is the input-output mutual information of an additive white Gaussian noise (AWGN) channel with input constellation $\mathcal{X}$ and received SNR ρ. We have that

$$I_{\mathcal{X}}(\rho) =$$
$$M - \frac{1}{2^M} \sum_{x \in \mathcal{X}} \mathbb{E}_Z \left[\log_2 \left(\sum_{x' \in \mathcal{X}} e^{-|\sqrt{\rho}(x-x')+Z|^2 + |Z|^2} \right) \right],$$

where the expectation is over $Z \sim \mathcal{N}_\mathbb{C}(0, 1)$. For communication at a fixed rate R, transmission is in outage whenever $I_B(\boldsymbol{p}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) < R$, and the corresponding outage probability is $\Pr(I_B(\boldsymbol{p}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) < R)$.

IV. POWER ALLOCATION FOR PEAK POWER CONSTRAINTS

A. Power Allocation Schemes

For systems with peak power constraint P_{peak}, the optimal power allocation scheme $\boldsymbol{p}^{\text{peak}}(\boldsymbol{\gamma})$ is given by

$$\boldsymbol{p}_{\text{peak}}(\boldsymbol{\gamma}) = \arg \min_{\substack{\boldsymbol{p} \succeq \boldsymbol{0} \\ \langle \boldsymbol{p} \rangle \leq P_{\text{peak}}}} \Pr(I_B(\boldsymbol{p}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) < R). \tag{3}$$

[2]Due to the perfect CSIR assumption, the phase can be compensated for.

A solution $\boldsymbol{p}^{\text{opt}}(\boldsymbol{\gamma})$ for the problem is given by

$$\boldsymbol{p}^{\text{opt}}(\boldsymbol{\gamma}) = \arg \max_{\substack{\boldsymbol{p} \succeq \boldsymbol{0} \\ \langle \boldsymbol{p} \rangle \leq P_{\text{peak}}}} I_B(\boldsymbol{p}, \boldsymbol{\gamma}). \tag{4}$$

From [5], we have that

$$p_b^{\text{opt}}(\boldsymbol{\gamma}) = \frac{1}{\gamma_b} \text{MMSE}_{\mathcal{X}}^{-1}\left(\min\left\{ 1, \frac{\eta}{\gamma_b} \right\} \right), b = 1, \ldots, B, \tag{5}$$

where $\text{MMSE}_{\mathcal{X}}(\rho)$ is the minimum mean-square error for estimating an input symbol in the constellation $\mathcal{X}$ transmitted over an AWGN channel with SNR ρ

$$\text{MMSE}_{\mathcal{X}}(\rho) = 1 - \frac{1}{\pi} \int_\mathbb{C} \frac{\left| \sum_{x \in \mathcal{X}} x e^{-|y-\sqrt{\rho}x|^2} \right|}{\sum_{x \in \mathcal{X}} e^{-|y-\sqrt{\rho}x|^2}} dy,$$

and η is chosen such that $\langle \boldsymbol{p}^{\text{opt}}(\boldsymbol{\gamma}) \rangle = P_{\text{peak}}$. We alternatively consider the following power allocation rule

$$\boldsymbol{p}_{\text{peak}}^{\text{opt}}(\boldsymbol{\gamma}) = \begin{cases} \wp^{\text{opt}}(\boldsymbol{\gamma}), & \langle \wp^{\text{opt}}(\boldsymbol{\gamma}) \rangle \leq P_{\text{peak}} \\ \boldsymbol{0}, & \text{otherwise}, \end{cases} \tag{6}$$

where $\wp(\boldsymbol{\gamma})$ solves the following problem

$$\wp^{\text{opt}}(\boldsymbol{\gamma}) = \arg \min_{\substack{\boldsymbol{p} \succeq \boldsymbol{0} \\ I_B(\boldsymbol{p}, \boldsymbol{\gamma}) \geq R}} \langle \boldsymbol{p} \rangle. \tag{7}$$

The solution of (7) is given by

$$\wp_b^{\text{opt}}(\boldsymbol{\gamma}) = \frac{1}{\gamma_b} \text{MMSE}_{\mathcal{X}}^{-1}\left(\min\left\{ 1, \frac{\eta}{\gamma_b} \right\} \right), b = 1, \ldots, B, \tag{8}$$

where η is now chosen such that $I_B(\wp^{\text{opt}}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) = R$.

The outage probability with peak power constraint P_{peak} is

$$P_{\text{out}}(\wp^{\text{opt}}(\boldsymbol{\gamma}), P_{\text{peak}}) = \Pr\left(\langle \wp^{\text{opt}}(\boldsymbol{\gamma}) \rangle > P_{\text{peak}} \right).$$

The power control rule given in (6) is also optimal, as given by the following Proposition.

Proposition 1: Consider transmission over the block-fading channel given in (1). The outage probability resulting from the power control rule given in (6) is the same as that resulting from the rule given in (5), i.e.

$$P_{\text{out}}(\wp^{\text{opt}}(\boldsymbol{\gamma}), P_{\text{peak}}) = \Pr(I_B(\boldsymbol{p}^{\text{opt}}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) < R), \tag{9}$$

for any fading statistics.

Proof: Assume that power control rule $\boldsymbol{p}^{\text{opt}}(\boldsymbol{\gamma})$ results in an outage for a channel realization $\boldsymbol{\gamma}$, i.e. $I_B(\boldsymbol{p}^{\text{opt}}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) < R$. Then, since $\boldsymbol{p}^{\text{opt}}(\boldsymbol{\gamma})$ is the solution of (4), for all power control rules $\wp(\boldsymbol{\gamma})$ satisfying $\langle \wp(\boldsymbol{\gamma}) \rangle \leq P_{\text{peak}}$, we have that $I_B(\wp(\boldsymbol{\gamma}), \boldsymbol{\gamma}) \leq I_B(\boldsymbol{p}^{\text{opt}}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) < R$. Consequently, since $I_B(\wp^{\text{opt}}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) \geq R$ due to the constraints in (7), we have $\langle \wp^{\text{opt}}(\boldsymbol{\gamma}) \rangle > P_{\text{peak}}$. Therefore, $\boldsymbol{p}_{\text{peak}}^{\text{opt}}(\boldsymbol{\gamma})$ also results in an outage for the channel realization $\boldsymbol{\gamma}$. With similar arguments, if a power control rule $\boldsymbol{p}_{\text{peak}}^{\text{opt}}(\boldsymbol{\gamma})$ results in an outage for channel realization $\boldsymbol{\gamma}$, then $\boldsymbol{p}^{\text{opt}}(\boldsymbol{\gamma})$ also gives an outage. Therefore, (9) follows. ∎

The optimal power allocation scheme in (5), (8) requires evaluating $\wp^{\text{opt}}(\boldsymbol{\gamma})$ for each channel realization $\boldsymbol{\gamma}$, which requires significant computational power or tabulating storage

capacity due to the involvement of the MMSE expression. This is not desirable in low cost devices. In [6], various suboptimal schemes are proposed, aiming at reducing the computational complexity. Among those, the truncated waterfilling scheme $\boldsymbol{p}^{\text{tw}}(\boldsymbol{\gamma})$ is given by

$$p_b^{\text{tw}}(\boldsymbol{\gamma}) = \min\left\{\frac{\beta}{\gamma_b}, \left(\eta - \frac{1}{\gamma_b}\right)\right\}, \ b = 1, \ldots, B, \quad (10)$$

where β is a predetermined parameter and η is chosen such that $\langle \boldsymbol{p}^{\text{tw}}(\boldsymbol{\gamma}) \rangle = P_{\text{peak}}$. Similar to the optimal case, we consider the following truncated waterfilling scheme:

$$\boldsymbol{p}_{\text{peak}}^{\text{tw}}(\boldsymbol{\gamma}) = \begin{cases} \wp^{\text{tw}}(\boldsymbol{\gamma}), & \text{if } \langle \wp^{\text{tw}}(\boldsymbol{\gamma}) \rangle \leq P_{\text{peak}} \\ \boldsymbol{0}, & \text{otherwise,} \end{cases} \quad (11)$$

where $\wp^{\text{tw}}(\boldsymbol{\gamma})$ is given by

$$\wp_b^{\text{tw}} = \min\left\{\frac{\beta}{\gamma}, \left(\eta - \frac{1}{\gamma_b}\right)\right\}, \ b = 1, \ldots, B, \quad (12)$$

with η chosen such that $I_B(\wp^{\text{tw}}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) = R$. The outage probability obtained by the scheme is similarly defined as

$$P_{\text{out}}(\wp^{\text{tw}}(\boldsymbol{\gamma}), P_{\text{peak}}) = \Pr(\langle \wp^{\text{tw}}(\boldsymbol{\gamma}) \rangle > P_{\text{peak}}).$$

The same duality as in Proposition 1 holds for the truncated waterfilling schemes. In particular, we have the following.

Proposition 2: Consider transmission over the block-fading channel given in (1). The outage probability resulting from the power control rule given in (11) is the same as that resulting from the rule given in (10), i.e.

$$P_{\text{out}}(\wp^{\text{tw}}(\boldsymbol{\gamma}), P_{\text{peak}}) = \Pr(I_B(\boldsymbol{p}^{\text{tw}}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) < R),$$

for any fading statistics.

Compared to the schemes given in (5), (10), those given in (6) and (11) are computationally more demanding, and are less practical for transmission with peak power constraints. For the optimal power control rules, while the outage probability obtained by power allocation schemes in (5) and (6) are the same, the power allocation scheme described in (6) is more useful in the analysis of systems with average power constraints. Similar observations apply for the truncated waterfilling schemes in (10) and (11). In the following analysis, we will refer to (6) and (11) for the optimal and truncated waterfilling schemes, respectively.

B. Asymptotic Performance

References [8], [7], [9], [10], [6] consider the asymptotic behavior of the outage probability. At high P_{peak}, under Nakagami-m fading statistics, the outage probability for the optimal power allocation scheme is given by [10], [6]

$$P_{\text{out}}(\wp^{\text{opt}}(\boldsymbol{\gamma}), P_{\text{peak}}) \doteq \mathcal{K}_{\text{peak}} P_{\text{peak}}^{-md_B(R)},$$

where $d_B(R)$ is the Singleton bound

$$d_B(R) = 1 + \left\lfloor B\left(1 - \frac{R}{M}\right)\right\rfloor.$$

Furthermore, for systems with truncated waterfilling schemes, we have the following result.

Proposition 3: Consider using the truncated waterfilling power control rule $\boldsymbol{p}_{\text{peak}}^{\text{tw}}$ described in (11) for transmission over the block-fading channel given in (1). Assume that the input constellation is $\mathcal{X}$. Further assume that the power fading gains follows the distribution in (2). For large P_{peak}, the outage probability behaves like

$$P_{\text{out}}(\wp^{\text{tw}}(\boldsymbol{\gamma}), P_{\text{peak}}) \doteq \mathcal{K}_{\text{peak}}^{\text{tw}} P_{\text{peak}}^{-md_\beta(R)}, \quad (13)$$

where $md_\beta(R)$ is the outage diversity, and

$$d_\beta(R) = 1 + \left\lfloor B\left(1 - \frac{R}{I_{\mathcal{X}}(\beta)}\right)\right\rfloor.$$

Proof: A sketch of the proof is as follows. From the characteristics of $\wp(\boldsymbol{\gamma})$, for all channel realization $\boldsymbol{\gamma}$, we have that $I_B(\boldsymbol{p}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) \geq I_{\mathcal{X}}(\beta) \sum_{b=1}^{B} \mathbb{1}_{\{P_{\text{peak}}\gamma_b \geq \beta\}}$. Moreover, from the power constraints $\langle \wp(\boldsymbol{\gamma}) \rangle \leq P_{\text{peak}}$, and from the fact that $\wp_b^{\text{tw}}\gamma_b \leq \beta$ (due to (12)), we have that $I_B(\boldsymbol{p}(\boldsymbol{\gamma}), \boldsymbol{\gamma}) \leq \sum_{b=1}^{B} I_{\mathcal{X}}^\beta(BP_{\text{peak}}\gamma_b)$ for all $\boldsymbol{\gamma}$, where $I_{\mathcal{X}}^\beta(\rho) \triangleq \min(I_{\mathcal{X}}(\beta), \log_2(1+\rho))$. Therefore,

$$P_{\text{out}}(\boldsymbol{p}_{\text{peak}}^{\text{tw}}(\boldsymbol{\gamma}), P_{\text{peak}}) \geq \Pr\left(\frac{1}{B}\sum_{b=1}^{B} I_{\mathcal{X}}^\beta(BP_{\text{peak}}\gamma_b) < R\right)$$

$$P_{\text{out}}(\boldsymbol{p}_{\text{peak}}^{\text{tw}}(\boldsymbol{\gamma}), P_{\text{peak}}) \leq \Pr\left(\frac{1}{B}\sum_{b=1}^{B} \mathbb{1}_{\{P_{\text{peak}}\gamma_b \geq \beta\}} < R\right).$$

Now, following the analysis in [10], we have that

$$\Pr\left(\frac{1}{B}\sum_{b=1}^{B} I_{\mathcal{X}}^\beta(BP_{\text{peak}}\gamma_b) < R\right) \doteq \mathcal{K}^\ell P_{\text{peak}}^{-d_\beta(R)}$$

$$\Pr\left(\frac{1}{B}\sum_{b=1}^{B} \mathbb{1}_{\{P_{\text{peak}}\gamma_b \geq \beta\}} < R\right) \doteq \mathcal{K}^u P_{\text{peak}}^{-d_\beta(R)},$$

leading to the asymptotic behavior in (13). $\blacksquare$

The asymptotic analysis given above is useful in characterizing the asymptotic performance of optimal and suboptimal power allocation schemes for systems with average power constraints, which is discussed in the next section.

V. POWER ALLOCATION FOR AVERAGE POWER CONSTRAINTS

A. Power Allocation Schemes

For systems with average power constraints, the optimal power allocation schemes is given by

$$\boldsymbol{p}_{\text{av}}(\boldsymbol{\gamma}) = \arg \min_{\substack{\boldsymbol{p} \succeq 0 \\ \mathbb{E}[\langle \boldsymbol{p} \rangle] \leq P_{\text{av}}}} P_{\text{out}}(\boldsymbol{p}(\boldsymbol{\gamma}), R). \quad (14)$$

Following [3], under Nakagami-m fading statistic, the solution of (14) has the following structure

$$\boldsymbol{p}_{\text{av}}(\boldsymbol{\gamma}) = \begin{cases} \wp(\boldsymbol{\gamma}), & \langle \wp(\boldsymbol{\gamma}) \rangle \leq s(\wp, P_{\text{av}}) \\ \boldsymbol{0}, & \text{otherwise,} \end{cases} \quad (15)$$

where $s(\wp(\boldsymbol{\gamma}), P_{\text{av}})$ is chosen such that

$$\begin{cases} s(\wp, P_{\text{av}}) = \infty, & \lim_{s \to \infty} \mathbb{E}\left[\langle \boldsymbol{p}_{\text{av}}(\boldsymbol{\gamma}) \rangle\right] < P_{\text{av}} \\ \mathbb{E}\left[\langle \boldsymbol{p}_{\text{av}}(\boldsymbol{\gamma}) \rangle\right] = P_{\text{av}}, & \text{otherwise.} \end{cases} \quad (16)$$

The solution of (14), which is also the optimal power allocation scheme $\boldsymbol{p}_{\mathrm{av}}^{\mathrm{opt}}(\boldsymbol{\gamma})$, is obtained by replacing $\wp(\boldsymbol{\gamma})$ in (15), (16) with $\wp^{\mathrm{opt}}(\boldsymbol{\gamma})$ given in (8). Similarly, the suboptimal truncated waterfilling scheme for systems with average power constraints $\boldsymbol{p}_{\mathrm{av}}^{\mathrm{tw}}(\boldsymbol{\gamma})$ is obtained by replacing $\wp(\boldsymbol{\gamma})$ with $\wp^{\mathrm{tw}}(\boldsymbol{\gamma})$ given in (11).

In general, the outage probability of the system employing the power allocation schemes given in (15) is $P_{\mathrm{out}}(\wp(\boldsymbol{\gamma}), s(\wp, P_{\mathrm{av}}))$, which is the outage probability of a system with power allocation scheme $\wp(\boldsymbol{\gamma})$ and peak power constraint $s(\wp, P_{\mathrm{av}})$. We employ this relationship, together with the knowledge of outage diversity of systems with peak power constraints, to characterize the outage diversity of systems with average power constraints.

B. Asymptotic Performance

For systems with average power constraints, we first characterize the relationship between P_{av} and s in (15). Given a threshold s, let $P_{\mathrm{av}}(s)$ be the average power required for transmission using the power control rule described by (15), the following theorem characterizes $P_{\mathrm{av}}(s)$.

Theorem 1: Assume a threshold s and a power allocation scheme described by (15). Further assuming that for large s,

$$P_{\mathrm{out}}(\wp(\boldsymbol{\gamma}), s) \doteq \mathcal{K}_{\mathrm{peak}} s^{-d(R)},$$

then, the resulting average power $P_{\mathrm{av}}(s) = \mathbb{E}\left[\langle \boldsymbol{p}^{\mathrm{av}}(\boldsymbol{\gamma})\rangle\right]$ satisfies,

$$\frac{d}{ds} P_{\mathrm{av}}(s) \doteq \mathcal{K}_{\mathrm{peak}} d(R) s^{-d(R)}. \tag{17}$$

Proof: A sketch of the proof is as follows. We have from definition that

$$\frac{d}{ds} P_{\mathrm{av}}(s) = \lim_{a \downarrow 1} \frac{P_{\mathrm{av}}(as) - P_{\mathrm{av}}(s)}{as - s}. \tag{18}$$

Let $\mathcal{R}(s, as) \triangleq \{\boldsymbol{\gamma} \in \mathbb{R}^B : s \leq \langle \wp(\boldsymbol{\gamma})\rangle \leq as\}$. From the power control rule given in (15),

$$\Delta P_{\mathrm{av}} \triangleq P_{\mathrm{av}}(as) - P_{\mathrm{av}}(s) = \int_{\boldsymbol{\gamma} \in \mathcal{R}(s, as)} \langle \wp(\boldsymbol{\gamma})\rangle f_{\boldsymbol{\gamma}}(\boldsymbol{\gamma}) d\boldsymbol{\gamma}. \tag{19}$$

Furthermore,

$$\int_{\boldsymbol{\gamma} \in \mathcal{R}(s, as)} f_{\boldsymbol{\gamma}}(\boldsymbol{\gamma}) d\boldsymbol{\gamma} = P_{\mathrm{out}}(\wp(\boldsymbol{\gamma}), s) - P_{\mathrm{out}}(\wp(\boldsymbol{\gamma}), as)$$
$$\doteq \mathcal{K}_{\mathrm{peak}} \left(s^{-d(R)} - (as)^{-d(R)}\right).$$

Therefore, noting the definition of $\mathcal{R}(s, as)$, (19) leads to

$$\Delta P_{\mathrm{av}} \dot{\leq} as \left(\mathcal{K}_{\mathrm{peak}} \left(s^{-d(R)} - (as)^{-d(R)}\right)\right) \tag{20}$$

$$\Delta P_{\mathrm{av}} \dot{\geq} s \left(\mathcal{K}_{\mathrm{peak}} \left(s^{-d(R)} - (as)^{-d(R)}\right)\right). \tag{21}$$

Finally, (17) follows by inserting the bounds in (20) and (21) into (18) and letting $a \downarrow 1$. ∎

The application of Theorem 1 to the asymptotic analysis of the outage probability with average power constraints leads to the following result.

Theorem 2: Consider transmission over the block-fading channel with a power allocation scheme $\boldsymbol{p}_{\mathrm{av}}(\boldsymbol{\gamma})$ given in (15). Assuming that for large s,

$$P_{\mathrm{out}}(\wp(\boldsymbol{\gamma}), s) \doteq \mathcal{K}_{\mathrm{peak}} s^{-d(R)},$$

then we have that
- If $d(R) > 1$, there exists a P_0 such that $s(\wp, P_0) = \infty$, or equivalently $P_{\mathrm{out}}(\wp(\boldsymbol{\gamma}), s(\wp, P_{\mathrm{av}})) = 0$ for $P_{\mathrm{av}} \geq P_0$. Therefore, the delay-limited capacity [12] is non-zero.
- If $d(R) < 1$, the asymptotic outage probability behaves as

$$P_{\mathrm{out}}(\wp(\boldsymbol{\gamma}), s(\wp, P_{\mathrm{av}})) \doteq \mathcal{K} P_{\mathrm{av}}^{-d_{\mathrm{av}}(R)},$$

where the average power outage diversity is related to the peak power outage diversity by

$$d_{\mathrm{av}}(R) = \frac{d(R)}{1 - d(R)}. \tag{22}$$

Proof: A sketch of the proof is as follows. For $d(R) > 1$, we have from Theorem 2 that

$$\lim_{s \to \infty} P_{\mathrm{av}}(s) = \int_0^\infty \frac{d}{ds} P_{\mathrm{av}}(s) ds$$
$$\doteq P(s_1) + \int_{s_1}^\infty \mathcal{K}_{\mathrm{peak}} s^{-d(R)} ds$$
$$\doteq P_0 < \infty.$$

Therefore, noting that $P_{\mathrm{av}}(s)$ is an increasing function of s, the first part of the theorem follows.

The outage diversity with respect to the average power is given by

$$d_{\mathrm{av}}(R) = \lim_{s \to \infty} \frac{-\log P_{\mathrm{out}}(\wp(\boldsymbol{\gamma}), s)}{\log P_{\mathrm{av}}(s)}$$
$$= \lim_{s \to \infty} \frac{-\log(\mathcal{K}_{\mathrm{peak}} s^{-d(R)})}{P_{\mathrm{av}}(s)}. \tag{23}$$

When $d(R) < 1$, $\lim_{s \to \infty} P_{\mathrm{av}}(s) = \infty$ and (22) is obtained by applying the L'Hôpital's rule to (23) and noting the result in Theorem 1. ∎

Theorems 1 and 2 characterize the asymptotic outage performance of power allocation schemes for systems with average power constraints given the asymptotic performance of the corresponding system with peak power constraints. The characterization is applicable to any fading statistics with continuous pdf. In particular, for systems employing the optimal power allocation schemes $\boldsymbol{p}^{\mathrm{opt}}(\boldsymbol{\gamma})$ over Nakagami-m faded channels, the outage diversity with peak power constraints is $m d_B(R)$. From Theorem 2, it follows that, if $m d_B(R) < 1$, the outage diversity of systems with average power constraints is given by $\frac{m d_B(R)}{1 - m d_B(R)}$, while if $m d_B(R) > 1$, the outage diversity is infinite and the outage probability curve is vertical. Therefore, delay-limited capacity R is achievable with a finite average transmit power P_R. Moreover, from Theorem 1, P_R can be approximated by

$$P_R = \lim_{s \to \infty} P_{\mathrm{av}}(s) \approx P_{\mathrm{av}}(s_1) + \int_{s_1}^\infty \frac{d}{ds} P_{\mathrm{av}}(s) ds$$
$$\approx P_{\mathrm{av}}(s_1) + \mathcal{K}_{\mathrm{peak}} \frac{m d_B(R)}{m d_B(R) - 1} s_1^{1 - m d_B(R)},$$

where s_1 is chosen such that the asymptotic analysis holds, and $P_{av}(s_1), \mathcal{K}_{peak}$ can be determined numerically. Similar conclusions can be drawn for systems employing an arbitrary power allocation scheme using the corresponding outage behavior at large peak power constraints. For example, asymptotic results for systems employing the truncated waterfilling scheme is obtained by replacing $d_B(R)$ in the previous analysis with $d_\beta(R)$.

The results are illustrated numerically in Figure 1 for transmission using the QPSK constellation over Nakagami-m block-fading channels with $B = 4$ and $m = 0.5, 2$ at rate $R = 1.7$. The figure shows the outage probability obtained by the optimal and the truncated waterfilling scheme with $\beta = 9$dB, respectively. In this case, the peak power constraint diversity is $d(R) = d_\beta(R) = d_B(R) = 1$. The figure shows that with average power constraints, outage diversity $\frac{md(R)}{1-md(R)} = 1$ is obtained when $md(R) = \frac{1}{2}$, while reliable transmission at rate R is possible (represented by the vertical slope of the two left most curves) when $md(R) = 2$.
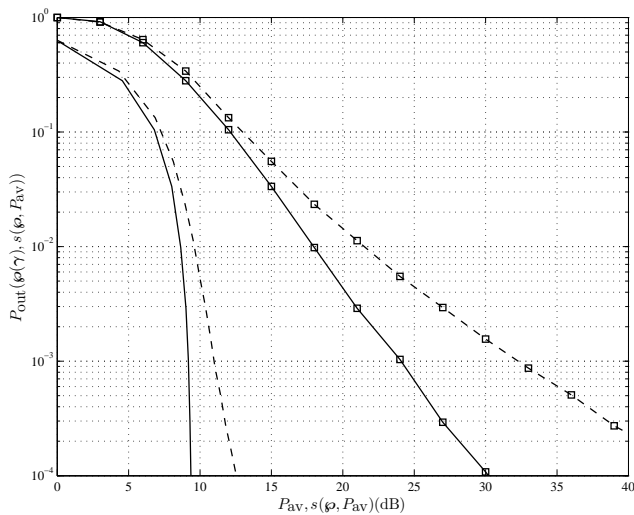

Fig. 1. Outage probability of transmissions with QPSK inputs over Nakagami-m block-fading channel with $B = 4, m = 0.5$ and $m = 2$. The solid lines and dashed lines correspondingly represent the optimal and the truncated waterfilling scheme. Lines with $\square$ are plots of outage probability vs. the peak power $s(\wp, P_{av})$ and lines without markers are plots of outage probability vs. the average power P_{av}.

Simulation results for the same setup with rate $R = 1.5, \beta = 7$dB and $m = 0.8$ are illustrated in Figure 2. In this case, $d_B(R) = 2$ and $d_\beta(R) = 1$, and therefore, we can observe the suboptimality in the outage diversity of systems employing the truncated waterfilling scheme[3]. For systems with average power constraints, the optimal power allocation scheme allows for reliable transmission at rate $R = 1.5$, while systems with the truncated waterfilling scheme have an outage diversity of $\frac{md_\beta(R)}{1-md_\beta(R)} = 4$.

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[3]Note that the β parameter is deliberately chosen to show the suboptimal outage diversity. In most cases, β can be chosen to obtain the optimal diversity. Otherwise, β can be chosen such that the suboptimal diversity does not appear at the outage levels of interest.
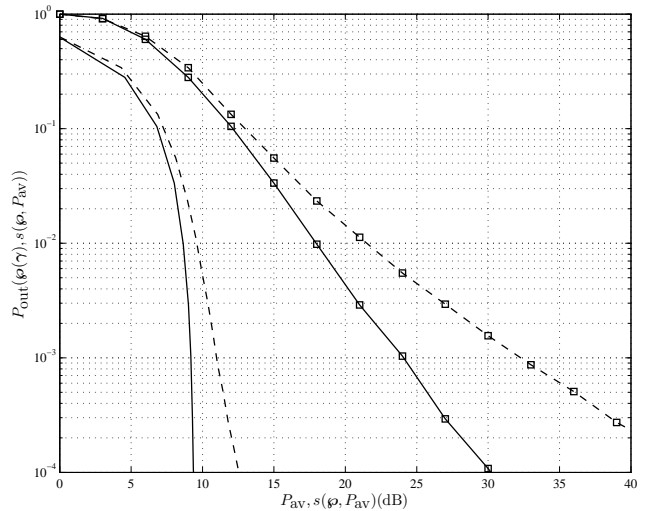

Fig. 2. Outage probability of transmissions with QPSK inputs over Nakagami-m block-fading channel with $B = 4, m = 0.8$. The solid lines and dashed lines correspondingly represent the optimal and the truncated waterfilling scheme. Lines with $\square$ are plots of outage probability vs. the peak power $s(\wp, P_{av})$ and lines without markers are plots of outage probability vs. the average power P_{av}.

VI. CONCLUSIONS

We have presented the asymptotic behavior of a power allocation rule for transmission at a target rate with average power constraints. We can determine the outage diversity, and thus the existence of positive delay-limited capacity, of an average-power-constraint system based on the diversity of the corresponding peak-power-constraint system. The delay-limited capacity can be approximated based on the asymptotic performance of the peak-power-constraint system.

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